Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education

MEI STRUCTURED MATHEMATICS   4768

Statistics 3

Thursday   **12 JANUARY 2006**   Afternoon   1 hour 30 minutes

Additional materials:
    8 page answer booklet
    Graph paper
    MEI Examination Formulae and Tables (MF2)

TIME   1 hour 30 minutes

INSTRUCTIONS TO CANDIDATES

- Write your name, centre number and candidate number in the spaces provided on the answer booklet.

- Answer **all** the questions.

- You are permitted to use a graphical calculator in this paper.

- Final answers should be given to a degree of accuracy appropriate to the context.

INFORMATION FOR CANDIDATES

- The number of marks is given in brackets [ ] at the end of each question or part question.

- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is being used.

- The total number of marks for this paper is 72.

This question paper consists of 5 printed pages and 3 blank pages.

1 A railway company is investigating operations at a junction where delays often occur. Delays (in minutes) are modelled by the random variable T with the following cumulative distribution function.

$$F(t) = \begin{cases} 0 & t \leqslant 0 \\ 1 - e^{-\frac{1}{3}t} & t > 0 \end{cases}$$

(i) Find the median delay and the 90th percentile delay. [5]

(ii) Derive the probability density function of T. Hence use calculus to find the mean delay. [5]

(iii) Find the probability that a delay lasts longer than the mean delay. [2]

You are given that the variance of T is 9.

(iv) Let $\overline{T}$ denote the mean of a random sample of 30 delays. Write down an approximation to the distribution of $\overline{T}$. [3]

(v) A random sample of 30 delays is found to have mean 4.2 minutes. Does this cast any doubt on the modelling? [3]

2   Geoffrey is a university lecturer. He has to prepare five questions for an examination. He knows by experience that it takes about 3 hours to prepare a question, and he models the time (in minutes) taken to prepare one by the Normally distributed random variable X with mean 180 and standard deviation 12, independently for all questions.

   (i) One morning, Geoffrey has a gap of 2 hours 50 minutes (170 minutes) between other activities. Find the probability that he can prepare a question in this time.   [3]

   (ii) One weekend, Geoffrey can devote 14 hours to preparing the complete examination paper. Find the probability that he can prepare all five questions in this time.   [3]

A colleague, Helen, has to check the questions.

   (iii) She models the time (in minutes) to check a question by the Normally distributed random variable Y with mean 50 and standard deviation 6, independently for all questions and independently of X. Find the probability that the total time for Geoffrey to prepare a question and Helen to check it exceeds 4 hours.   [3]

   (iv) When working under pressure of deadlines, Helen models the time to check a question in a different way. She uses the Normally distributed random variable $\frac{1}{4}X$, where X is as above. Find the length of time, as given by this model, which Helen needs to ensure that, with probability 0.9, she has time to check a question.   [4]

Ian, an educational researcher, suggests that a better model for the time taken to prepare a question would be a constant k representing "thinking time" plus a random variable T representing the time required to write the question itself, independently for all questions.

   (v) Taking k as 45 and T as Normally distributed with mean 120 and standard deviation 10 (all units are minutes), find the probability according to Ian's model that a question can be prepared in less than 2 hours 30 minutes.   [2]

Juliet, an administrator, proposes that the examination should be reduced in time and shorter questions should be used.

   (vi) Juliet suggests that Ian's model should be used for the time taken to prepare such shorter questions but with $k = 30$ and T replaced by $\frac{3}{5}T$. Find the probability as given by this model that a question can be prepared in less than $1\frac{3}{4}$ hours.   [3]

    [Turn over

3 A production line has two machines, A and B, for delivering liquid soap into bottles. Each machine is set to deliver a nominal amount of 250 ml, but it is not expected that they will work to a high level of accuracy. In particular, it is known that the ambient temperature affects the rate of flow of the liquid and leads to variation in the amounts delivered.

The operators think that machine B tends to deliver a somewhat greater amount than machine A, no matter what the ambient temperature. This is being investigated by an experiment. A random sample of 10 results from the experiment is shown below. Each column of data is for a different ambient temperature.

Ambient temperature	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9	T_{10}
Amount delivered by machine A	246.2	251.6	252.0	246.6	258.4	251.0	247.5	247.1	248.1	253.4
Amount delivered by machine B	248.3	252.6	252.8	247.2	258.8	250.0	247.2	247.9	249.0	254.5

(a) Use an appropriate *t* test to examine, at the 5% level of significance, whether the mean amount delivered by machine B may be taken as being greater than that delivered by machine A, stating carefully your null and alternative hypotheses and the required distributional assumption. [11]

(b) Using the data for machine A in the table above, provide a two-sided 95% confidence interval for the mean amount delivered by this machine, stating the required distributional assumption. Explain whether you would conclude that the machine appears to be working correctly in terms of the nominal amount as set. [7]

4 Quality control inspectors in a factory are investigating the lengths of glass tubes that will be used to make laboratory equipment.

  (i) Data on the observed lengths of a random sample of 200 glass tubes from one batch are available in the form of a frequency distribution as follows.

Length x (mm)	Observed frequency
$x \leqslant 298$	1
$298 < x \leqslant 300$	30
$300 < x \leqslant 301$	62
$301 < x \leqslant 302$	70
$302 < x \leqslant 304$	34
$x > 304$	3

The sample mean and standard deviation are 301.08 and 1.2655 respectively.

The corresponding expected frequencies for the Normal distribution with parameters estimated by the sample statistics are

Length x (mm)	Expected frequency
$x \leqslant 298$	1.49
$298 < x \leqslant 300$	37.85
$300 < x \leqslant 301$	55.62
$301 < x \leqslant 302$	58.32
$302 < x \leqslant 304$	44.62
$x > 304$	2.10

Examine the goodness of fit of a Normal distribution, using a 5% significance level.   [7]

 (ii) It is thought that the lengths of tubes in another batch have an underlying distribution similar to that for the batch in part **(i)** but possibly with different location and dispersion parameters. A random sample of 10 tubes from this batch gives the following lengths (in mm).

    301.3   301.4   299.6   302.2   300.3   303.2   302.6   301.8   300.9   300.8

    (A) Discuss briefly whether it would be appropriate to use a t test to examine a hypothesis about the population mean length for this batch.   [2]

    (B) Use a Wilcoxon test to examine at the 10% significance level whether the population median length for this batch is 301 mm.   [9]

Mark Scheme 4768
January 2006

Q1	$F(t) = 1 - e^{-t/3}$ (t>0)			
(i)	For median m, $\frac{1}{2} = 1 - e^{-m/3}$	M1		
	$\therefore e^{-m/3} = \frac{1}{2} \Rightarrow -\frac{m}{3} = \ln\frac{1}{2} = -0.6931$	M1	attempt to solve, here or for 90th percentile. Depends on previous M mark.	
	$\Rightarrow m = 2.079$	A1		
	For 90th percentile p, $0.9 = 1 - e^{-p/3}$	M1		
	$\therefore e^{-p/3} = 0.1 \Rightarrow -\frac{p}{3} = \ln 0.1 = -2.3026$			
	$\Rightarrow p = 6.908$	A1		5
(ii)	$f(t) = \frac{d}{dt} F(t)$	M1		
	$= \frac{1}{3} e^{-t/3}$	A1	(for t>0, but condone absence of this)	
	$\mu = \int_0^\infty \frac{1}{3} t e^{-t/3} dt$	M1	Quoting standard result gets 0/3 for the mean.	
	$= \frac{1}{3}\left\{ \left[\frac{t e^{-t/3}}{-1/3} \right]_0^\infty + 3\int_0^\infty e^{-t/3} dt \right\}$	M1	attempt to integrate by parts	
	$= [0 - 0] + \left[\frac{e^{-t/3}}{-1/3} \right]_0^\infty = 3$	A1		5
(iii)	$P(T > \mu) = [\text{from cdf}]\; e^{-\mu/3} = e^{-1}$	M1	[or via pdf]	
	$= 0.3679$	A1	ft c's mean (>0)	2
(iv)	$\bar{T} \sim (\text{approx}) \; N\left(3, \frac{9}{30} = 0.3\right)$	B1	N	
		B1	ft c's mean (>0)	
		B1	0.3	3
(v)	**EITHER** can argue that 4.2 is more than 2 SDs from μ	M1		
	($3 + 2\sqrt{0.3} = 4.095$;			
	<u>must</u> refer to SD ($\bar{T}$), not SD(T))			
	i.e. outlier	M1		
	$\Rightarrow$ doubt	A1		3
	OR formal significance test:	M1		
	$\frac{4.2 - 3}{3/\sqrt{30}} = 2.191$, refer to N(0,1), sig at (eg) 5%	M1	Depends on first M, but could imply it.	
	$\Rightarrow$ doubt	A1		
				18

Q2	$X \sim \text{N}(180, \sigma = 12)$		When a candidate's answers suggest that (s)he appears to have neglected to use the difference columns of the Normal distribution tables penalise the first occurrence only.	
(i)	$\text{P}(X < 170) = \text{P}\left(Z < \dfrac{170-180}{12} = -0.8333\right)$ $= 1 - 0.7976 = 0.2024$	M A1 A1	For standardising. Award once, here or elsewhere.	3
(ii)	$X_1 + X_2 + X_3 + X_4 + X_5 \sim \text{N}\left(900, \sigma^2 = 720[\sigma = 26.8328]\right)$ $\text{P}(\text{this} < 840) = \text{P}\left(Z < \dfrac{840-900}{26.8328} = -2.236\right)$ $= 1 - 0.9873 = 0.0127$	B1 B1 A1	Mean. Variance. Accept sd. c.a.o.	3
(iii)	$Y \sim \text{N}(50, \sigma = 6)$ $X + Y \sim \text{N}(230, \sigma^2 = 180[\sigma = 13.4164])$ $\text{P}(\text{this} > 240) = \text{P}\left(Z > \dfrac{240-230}{13.4164} = 0.7454\right)$ $= 1 - 0.7720 = 0.2280$	B1 B1 A1	Mean. Variance. Accept sd. c.a.o.	3
(iv)	$\dfrac{1}{4}X \sim N\left(45, \sigma^2 = \dfrac{1}{16} \times 144 = 9[\sigma = 3]\right)$ Require t such that $0.9 = \text{P}(\text{this} < t) = \text{P}\left(Z < \dfrac{t-45}{3}\right) = \text{P}(Z < 1.282)$ $\therefore t - 45 = 3 \times 1.282 \Rightarrow t = 48.85 \ (48.846)$	B1 M B1 A1	Variance. Accept sd. FT incorrect mean. Formulation of requirement. 1.282 ft only for incorrect mean	4
(v)	$I = 45 + T$ where $T \sim \text{N}(120, \sigma = 10)$ $\therefore I \sim \text{N}(165, \sigma = 10)$ $\text{P}(I < 150) = \text{P}\left(Z < \dfrac{150-165}{10} = -1.5\right)$ $= 1 - 0.9332 = 0.0668$	B1 A1	for unchanged σ (candidates might work with P($T<105$)) c.a.o.	2
(vi)	$J = 30 + \dfrac{3}{5}T$ where $T \sim \text{N}(120, \sigma = 10)$		Cands might work with $\text{P}\left(\tfrac{3}{5}T < 75\right)$. $\tfrac{3}{5}T \sim N(72, 36)$	

	$\therefore J \sim N\left(102, \sigma^2 = \dfrac{9}{25} \times 100 = 36 \left[\sigma = 6\right]\right)$	B1 B1	Mean. Variance. Accept sd.	
	$P(J < 105) = P(Z < \dfrac{105 - 102}{6} = 0.5) = 0.6915$	A1	c.a.o.	3
				18

Q3				
(a)	H_0: $\mu_D = 0$ (or $\mu_A = \mu_B$)	B1	Hypotheses in words only must include "population".	
	H_1: $\mu_D > 0$ (or $\mu_B > \mu_A$) where μ_D is "mean for B – mean for A"	B1 B1	Or "<" for $A - B$. For adequate verbal definition. Allow absence of "population" if correct notation μ is used, but do NOT allow "$\overline{X}_A = \overline{X}_B$" or similar unless $\overline{X}$ is clearly and explicitly stated to be a population mean.	
	Normality of <u>differences</u> is required <u>MUST</u> be PAIRED COMPARISON t test. Differences are: 2.1 1.0 0.8 0.6 0.4 -1.0 -0.3 0.8 0.9 1.1	B1		
	$\bar{d} = 0.64$ $s_{n-1} = 0.8316$	B1	$s_n = 0.7889$ but do <u>NOT</u> allow this here or in construction of test statistic, but FT from there.	
	Test statistic is $\dfrac{0.64 - 0}{\dfrac{0.8316}{\sqrt{10}}}$	M	Allow c's $\bar{d}$ and/or s_{n-1}. Allow alternative: $0 + (\text{c's } 1.833) \times \frac{0.8316}{\sqrt{10}}$ $(= 0.4821)$ for subsequent comparison with $\bar{d}$. (Or $\bar{d} - (\text{c's } 1.833) \times \frac{0.8316}{\sqrt{10}}$ $(= 0.1579)$ for comparison with 0.)	
	=2.43(37).	A1	c.a.o. but ft from here in any case if wrong. Use of $0 - \bar{d}$ scores M1A0, but ft.	
	Refer to t_9.	M	No ft from here if wrong.	
	Single-tailed 5% point is 1.833. Significant. Seems mean amount delivered by B is greater that that by A	A1 E1 E1	No ft from here if wrong. ft only c's test statistic. ft only c's test statistic. Special case: (t_{10} and 1.812) can score 1 of these last 2 marks if either form of conclusion is given.	11
(b)	We now require Normality for the amounts delivered by machine A.	B1		

For machine A, $\bar{x} = 250.19$ $s_{n-1} = 3.8527$	B1	$s_n = 3.6549(83)$ but do NOT allow this here or in construction of CI.	
CI is given by $250.19 \pm 2.262\dfrac{3.8527}{\sqrt{10}}$	M	ft c's $\bar{x} \pm$. 2.262	
	B1 M	ft c's s_{n-1}.	
$= 250.19 \pm 2.75(6) = (247.43(4),$ 252.94(6))	A1	c.a.o. Must be expressed as an interval. ZERO/4 if not same distribution as test. Same wrong distribution scores maximum M1B0M1A0. Recovery to t_9 is OK.	
250 is in the CI, so would accept H$_0$: $\mu =$ 250, so no evidence that machine is not working correctly in this respect.	E1		7
			18

Q4				
(i)				

	1 ⎵ 30	62	70	34 ⎵ 3
	31			37
e_i	1.49 ⎵ 37.85	55.62	58.32	44.62 ⎵ 2.10
	39.34			46.72

	M	for grouping	
$X^2 = 1.7681 + 0.7318 + 2.3392 + 2.0222$	M	Allow the M1 for correct method from wrongly grouped or ungrouped table.	
$= 6.86$	A1		
Refer to χ_1^2.	M	Allow correct df (= cells – 3) from wrongly grouped or ungrouped table, and FT. Otherwise, no FT if wrong.	
Upper 5% point is 3.84	A1	No ft from here if wrong.	
Significant	E1	ft only c's test statistic.	
Suggests Normal model does not fit	E1	ft only c's test statistic.	7
(ii) (A)	t test unwise …	E1		
	… because underlying population appears non-Normal	E1	FT from result of candidate's work in (i)	2

(B)

Data	Median 301	Difference	Rank of |diff|		
301.3		0.3	3	M	for differences. ZERO in this section if differences not used.
301.4		0.4	4		
299.6		- 1.4	8		
302.2		1.2	7		
300.3		- 0.7	5		for ranks.
303.2		2.2	10	M	FT if ranks wrong.
302.6		1.6	9		
301.8		0.8	6	A1	
300.9		- 0.1	1		
300.8		- 0.2	2		

$T = 1 + 2 + 5 + 8 = 16$ (or $3+4+6+7+9+10 = 39$)   **B1**

Refer to tables of Wilcoxon single sample (/paired) statistic   **M**

Lower (or upper if 39 used) 5% tail is needed   **M**

Value for $n = 10$ is 10 (or 45 if 39 used)   **A1**

Result is not significant   **E1**

No evidence against median being 301   **E1**   **9**

| | | | | | **18** |

4768: Statistics 3

General Comments

The overall standard of the scripts seen was pleasing: many candidates appeared well prepared for this paper. However, the quality of their comments, interpretations and explanations was consistently below that of the rest of the work.

Invariably all four questions were attempted. Question 2 was found to be very accessible and most candidates scored full or nearly full marks. Questions 1, 3 and 4 were well answered, with many candidates scoring relatively high marks. There was no evidence to suggest that candidates found themselves short of time at the end.

Comments on Individual Questions

1) **Continuous random variables; exponential distribution; Central Limit Theorem; delays at a railway junction.**
    (i)   This part of the question was almost always answered correctly.
    (ii)   Most candidates found the pdf correctly, though occasional errors with the differentiation of the exponential function were seen.
            By contrast quite a few candidates got into difficulty over finding the mean delay, often because they did not seem able to handle the integration by parts successfully. There were many candidates who sailed through this part with apparent ease.
    (iii)   By and large candidates seemed to be aware that in order to answer this part they needed to use the cdf and work out $1 - F(\mu)$.
    (iv)   This part of the question was not answered as well as might have been expected. There were some who thought that the mean of the distribution of $\bar{T}$ would be $\bar{T}$ itself, while others multiplied the mean and variance by 30.
    (v)   It was expected that candidates would use the distribution of $\bar{T}$ from the previous part either to carry out a brief, informal hypothesis test or to consider the usual criterion (mean ± 2 s.d.'s) for an outlier. In practice many candidates' answers fell somewhere between the two, which was quite acceptable.

2) **Combinations of Normal distributions; the times taken to set and check exam questions.**
    (i)   This part was almost always answered correctly.
    (ii)   A few candidates worked out the variance here incorrectly as $5^2 \times 12^2$. Other than that, this part was usually correct.
    (iii)   Only a few candidates experienced difficulty in getting this part right.
    (iv)   Candidates managed to get this part wrong in one of two ways. Either they worked out the variance as $12^2 \div 4$, or they found and used the distribution of ¼Y (from part (iii)), believing it to be ¼X. The value of $\Phi^{-1}(0.9)$ was almost always quoted and used correctly.
    (v)   Candidates seemed to have little difficulty with this part. Most preferred to work with P(T<105).
    (vi)   In this part most candidates preferred to work with P($3T/5$<75), but some worked out the variance incorrectly as 60 rather than 36.

3)   **The _t_ distribution: paired hypothesis test for the difference between two population means; confidence interval for a population mean; production lines filling bottles of liquid soap.**

(a)   The hypotheses were usually stated satisfactorily but many candidates did not define their symbol μ adequately as the mean _difference_. Nor did they manage to convey that the required assumption was that the _differences_ come from a Normal population.

The test statistic was usually worked out correctly from the data. Very few candidates failed to recognise that this was a paired comparison. The correct _t_ distribution was usually used for the test, and the only blemish was in the final conclusion: candidates did not take care to express this carefully enough, including mentioning that it is the mean that has been tested.

(b)   By and large the confidence interval was found correctly, but there was a smattering of the types of errors that one might anticipate such as the wrong percentage point. Many candidates gave a satisfactory explanation about the correct working of the machine with regard to the nominal setting, although quite a few felt it necessary to quote a generic interpretation of a confidence interval. Candidates needed to take care over the required assumption here. In a question like this, including part (a) above, it is important to be clear about which population is required to be Normal.

4)   **Chi-squared hypothesis test for the goodness of fit of a Normal model; Wilcoxon test for a population median; manufacture of glass tubes.**

(i)   There was some uncertainty about the need for combining classes in this question. Nonetheless the correct test statistic was obtained in the majority of cases.

Many candidates stated the number of degrees of freedom incorrectly and hence looked up the wrong critical value. This was usually because they did not allow for the estimated parameters (the mean and variance) and/or for having combined classes.

(ii)

(_A_)   It appeared that a lot of candidates did not read this part of the question carefully enough. They may have known that a Normal population is required for a _t_ test, but they certainly did not appreciate the significance of referring back to their conclusion to part (i) for guidance.

(_B_)   The Wilcoxon test was carried out very competently by many candidates. Occasionally there was some carelessness in working out the differences from the stated median and this resulted in incorrect rankings. Sometimes the final part of the conclusion, which is expected to be in context, was expressed poorly.